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THE PERIODIC PROBLEM OF REINFORCEMENT OF A PLATE WEAKENED BY A SYSTEM OF CUTS USING STRINGERS

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A plate containing a system of cuts is considered. The cuts are oriented along a straight line, and the plate is reinforced with a periodic set of stiffening ribs (stringers) in direction perpendicular to that line. A tensile force is applied to the plate in the direction perpendicular to the line of cuts. The problem is reduced to a system of singular, integro-differential equations. The results of the computations are presented in the form of graphs characterizing the dependence of the stress intensities at the ends of the cuts and stringers, on their length and on the rigidity of the stringers. The problem of the effect of the stringers on the state of stress of a weakened plate was studied by a number of authors. In particular, the combination of a stringer and a circular hole was dealt with in [1], two stringers situated symmetrically about a circular hole were considered in [2], a stringer and a crack in [1, 3], etc. The methods developed in these works can be combined with the methods of solving the problems of the mathematical theory of cracks [4] to provide an effective way if investigating a periodic system of custs [5] strengthened by a periodic system of stiffening ribs.

The aim of this paper is to estimate the effect of stringers orthogonal to the line along which the periodic cuts are distributed, on the stress intensity coefficients at the ends of these custs.

Let a plate be given containing a periodic system of cuts and a periodic system of stringers (Fig. 1). The custs lie along the straight line $y = 0$, are of equal length of $2c$ and are situated with the intervals of length $2b$ $(c < b)$ so that their middle points $x_k = (2k + 1)b$ $(k = 0, \pm 1, \pm 2, \ldots)$ coincide with the centers ot the intervals. The stringers which are of equal length $2a$ $(a < b)$ and continuously attached to the plate, pass through the ends of the intervals $x_k = 2kb$ and are perpendicular to the straight line $y = 0$. The stringers are free to bend, and work only under tension. E, ν and h denote the modulus of elasticity, the Poisson's ratio and the thickness of the plate respectively, while E_0 and S_0 are the modulus of

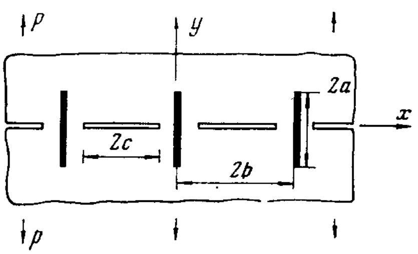

Fig. 1

elasticity and the area of transverse cross section of the stringer.

The following notation is adopted for the elements of the elastic fields: σ_x, σ_y and τ_{xy} are the stress components, u and v are the plate displacement components, $N(y)$ is the normal force in the cross section of the stringer and $\varepsilon^\circ(y)$ denotes the relative elongation of its axis. The following tensile forces act upon the plate:

$$\sigma_y^\infty = p = \text{const}, \quad \sigma_x^\infty = \tau_{xy}^\infty = 0 \tag{1}$$

and the contours of the cuts are stress free.

Let us quote the relations [1] defining the problem. The conditions of equilibrium of any infinitesimal element of the stringer $L_k = \{x = 2kb, \ |y| < a\}$ attached to the plate along its whole length, of the absence of resistance to bending within the stringer and of the continuity of the displacement components and of relative elongation $\varepsilon_y = \partial v / \partial y$ on the passage across the axis of the stringer, have the form

$$h(\tau_{xy}^+ - \tau_{xy}^-) - N'(y) = 0, \quad \sigma_x^+ - \sigma_x^- = 0 \tag{2}$$

$$u^+ + iv^+ = u^- + iv^-, \quad \varepsilon_y^+ = \varepsilon_y^- = \varepsilon^\circ \tag{3}$$

The expressions (2) together with the formula $N(y) = E_0 S_0 \varepsilon^\circ = E_0 S_0 v'^+$, yield

$$h\int_{-a}^{y} [(\sigma_x + i\tau_{xy})^+ - (\sigma_x + i\tau_{xy})^-]\,dy - iE_0S_0\left(\frac{\partial v}{\partial y}\right)^+ = 0 \tag{4}$$

The conditions of absence of the normal and tangential stresses at the cut edges $l_k = \{|(2k+1)b - x| < c, \ y = 0\}$ have the form

$$\sigma_y^+ + i\tau_{xy}^+ = \sigma_y^- + i\tau_{xy}^- = 0 \tag{5}$$

Let us introduce the Kolosov—Muskhelishvili functions $\Phi(z)$ and $\Psi(z)$. According to [6] we have

$$\begin{aligned} &\sigma_x + \sigma_y = 2[\Phi(z) + \overline{\Phi(z)}] \\ &\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar z\Phi'(z) + \Psi(z)] \\ &2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \\ &\Phi(z) = \varphi'(z), \quad \Psi(z) = \psi'(z), \quad z = x + iy \\ &2\mu = E/(1+\nu), \quad \varkappa = (3-\nu)/(1+\nu) \end{aligned} \tag{6}$$

Then the relations (3)—(5) transform into the following boundary value problem:

$$\begin{aligned} &H_1^+(t_1) - H_1^-(t_1) = 0, \quad t_1 \in L_k \\ &(\varkappa+1)[\varphi^+(t_1) - \varphi^-(t_1)] + \lambda_0 \operatorname{Re} H_2^+(t_1) = 0 \\ &H_3^\pm(t_2) = 0, \quad t_2 \in l_k \end{aligned} \tag{7}$$

where

$$\begin{aligned} &H_1(z) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \\ &H_2(z) = \varkappa\varphi'(z) - \overline{\varphi'(z)} + z\overline{\varphi''(z)} + \overline{\psi'(z)} \\ &H_3(z) = \varphi'(z) + \overline{\varphi'(z)} + \bar z\varphi''(z) + \psi'(z) \\ &\lambda_0 = E_0 s_0/(2\mu h) \end{aligned}$$

Let us assume

$$\Phi(z) = \sum_{k=-\infty}^{\infty} [\Phi_{1k}(z) + \Phi_{2k}(z)] + \frac{p}{4h} \tag{8}$$

$$\Psi(z) = \sum_{k=-\infty}^{\infty} \{\Phi_{3k}(z) - [z\Phi_{1k}(z)]' + \Phi_{4k}(z) +$$
$$[(z - 4kb)\,\Phi_{2k}(z)]'\} + \frac{p}{2h}$$
$$\Phi_{jk}(z) = \frac{1}{2\pi i}\int_{b_{jk}} \frac{f_j(\zeta)\,d\zeta}{\zeta - z}, \quad j = 1,3$$
$$\varphi_{jk}(z) = \frac{1}{2\pi i}\int_{b_{jk}} \frac{f_j(\zeta)\,d\zeta}{\zeta - z}, \quad \Phi_{jk}(z) = \varphi_{jk}(z), \quad j = 2,4$$
$$b_{1k} = b_{3k} = \{(2k+1)\,b - c,\ (2k+1)\,b + c\}$$
$$b_{2k} = b_{4k} = \{2kb - ia,\ 2kb + ia\}$$

The above representations allow the conditions (1) to be fulfilled. The functions $\Phi_{1k}(z)$, $\Phi_{3k}(z)$ become discontinuous during the passage across the cut, and $\varphi_{2k}(z)$, $\varphi_{4k}(z)$ during the passage across the line of the stringer. The second and fourth relations of (8) hold, if $f_3(\zeta) = -\overline{f_1(\zeta)}$, $f_4(\zeta) = \varkappa f_2(\zeta)$.

For the remaining two functions we obtain a system of singular, integro-differential equations which, after separating the real and imaginary parts ($f_j = \alpha_j + i\beta_j$, $j = 1, 2$) and using the expansions (see [7]), of trigonometric and hyperbolic functions into the sums of partial fractions, assumes the form

$$\beta_2(y) = 0 \tag{9}$$
$$\int_{b-c}^{b+c} W_1(\xi, x)\,\beta_1(\xi)\,d\xi + \int_{-a}^{a} W_3(\xi, x)\,\alpha_2(\xi)\,d\xi + \frac{\pi p}{h} = 0$$
$$\int_{b-c}^{b+c} W_1(\xi, x)\,\alpha_1(\xi)\,d\xi + \int_{-a}^{a} W_4(\xi, x)\,\alpha_2(\xi)\,d\xi = 0$$
$$\pi(\varkappa + 1)\,\alpha_2(y) + \lambda_0\left\{\int_{-a}^{a} W_2(\xi, y)\,\alpha_2'(\xi)\,d\xi + \right.$$
$$\left.\int_{b-c}^{b+c} [W_5(\xi, y)\,\alpha_1(\xi) + W_6(\xi, y)\,\beta_1(\xi)]\,d\xi + (\varkappa + 1)\frac{p}{4h}\right\} = 0$$

where

$$W_1 = \omega \operatorname{ctg} \omega(\xi - x - b),$$
$$W_2 = -\omega\left[\varkappa - 1 + (\xi - y)\frac{d}{dy}\right]\operatorname{ctg}\omega(\xi - y)$$
$$W_3 = \frac{d}{dx}\left[(\varkappa + 1)\,W_7 + 2\xi\frac{d}{dx}W_8\right]$$
$$W_4 = \frac{d}{dx}\left[(\varkappa + 3)\,W_8 - 2\xi\frac{d}{dx}W_7\right]$$
$$W_5 = \frac{\omega}{4}\left[\varkappa - 3 + 2y\frac{d}{dy}\right](\gamma_1^{-1}\operatorname{sh} 2\omega y)$$
$$W_6 = \frac{\omega}{4}\left[\varkappa - 1 + 2y\frac{d}{dy}\right](\gamma_1^{-1}\sin 2\omega\xi)$$
$$W_7 = -\frac{\omega}{4}\gamma_2^{-1}\sin 2\omega x, \quad W_8 = \frac{\omega}{4}\gamma_2^{-1}\operatorname{sh} 2\omega\xi$$
$$\gamma_1 = \sin^2\omega\xi + \operatorname{sh}^2\omega y, \quad \gamma_2 = \cos^2\omega x + \operatorname{sh}^2\omega\xi, \quad \omega = \frac{\pi}{2b}$$

The second equation of the system (9) holds identically, provided that we set

$$\alpha_1(\xi)=0, \quad \beta_1(b-\xi)=-\beta_1(b+\xi), \quad \alpha_2(-\xi)=\alpha_2(\xi)$$

Let us introduce the following change of variables:

$$x=cx^*, \quad \xi=b+c\tau, \quad x,\xi\in(b-c,\,b+c)$$
$$y=ay^*, \quad \xi=a\tau, \quad y,\xi\in(-a,\,a)$$
$$\beta_1(\xi)=\frac{p}{h}g_1(\tau), \quad \alpha_2(\xi)=\frac{p}{h}g_2(\tau)$$

This yields the following system of equations (with the asterisk omitted):

$$\frac{1}{\pi}\int_{-1}^{1}\{Q_1(\tau,x)g_1(\tau)+Q_2(\tau,x)g_2(\tau)\}\,d\tau=-1, \quad |x|<1 \tag{10}$$

$$g_2(y)-\frac{\lambda\varkappa}{\pi}\int_{-1}^{1}\frac{g_2'(\tau)\,d\tau}{\tau-y}+$$
$$\frac{1}{\pi}\int_{-1}^{1}\{Q_3(\tau,y)g_2(\tau)+Q_4(\tau,y)g_1(\tau)\}\,d\tau=-\frac{\lambda_0}{4\pi}, \quad |y|<1$$

where

$$Q_1(\tau,x)=e\,\mathrm{ctg}\,e(\tau-x)$$
$$Q_2(\tau,x)=-\frac{s^2}{4e}\frac{d}{dx}\Big[(\varkappa+1)(\sin 2exh(x,\tau))-$$
$$\frac{2s\tau}{e}\frac{d}{dx}(\mathrm{sh}\,2s\tau h(x,\tau))\Big]$$
$$Q_3(\tau,y)=-\frac{s^2}{3}(\varkappa-2)+\frac{s^4}{15}(\varkappa-4)(\tau-y)^2-$$
$$\frac{2s^6}{189}(\varkappa-6)(\tau-y)^4+O(s^8)$$
$$Q_4(\tau,y)=-\frac{e}{2}\Big[\varkappa-1+2y\frac{d}{dy}\Big](\sin 2e\tau h(\tau,y))$$
$$h(x,y)=(\mathrm{ch}^2 sy-\sin^2 ex)^{-1}$$
$$e=c\omega, \quad s=a\omega, \quad \lambda=\lambda_0[a(1+\varkappa)]^{-1}=(1+\nu)^2 S_0E_0/(2ahE)$$

The first equation of (10) can be solved for $g_1(x)$. Indeed, consider the finctions

$$F_1(z)=\frac{e}{2\pi}\int_{-1}^{1}g_1(\tau)\,\mathrm{ctg}\,e(\tau-z)\,d\tau \tag{11}$$

$$F_2(z)=\frac{1}{R(z)}\frac{e}{2\pi}\int_{-1}^{1}\frac{\gamma(\tau)R^+(\tau)\cos e\tau}{\sin e\tau-\sin ez}\,d\tau+\frac{1}{2}\Big[\frac{\sin ez}{R(z)}-1\Big]$$
$$(R(z)=(\sin^2 ez-\sin^2 e)^{1/2}, \quad R^+(\tau)=(\sin^2 e-\sin^2 e\tau)^{1/2})$$

Using the Sokhotskii formulas we can confirm that at the cuts we have, by virtue of (10),

$$F_1^+(x)+F_1^-(x)=F_2^+(x)+F_2^-(x), \quad x\in l_k$$

and this yields (see [6], Sect. 108) $F_1(z)\equiv F_2(z)$. Then

$$F_1^+(x)-F_1^-(x)=F_2^+(x)-F_2^-(x), \quad x\in l_k$$

or

$$g_1(x) = -[R^+(x)]^{-1}\left(\sin ex + \frac{e}{\pi}\int_{-1}^{1}\frac{\gamma(\tau)R^+(\tau)\cos e\tau}{\sin e\tau - \sin ex}\,d\tau\right) \tag{12}$$

$$\left(\int_{-1}^{1} g_1(x)\,dx = 0\right)$$

$$\gamma(x) = -\frac{1}{\pi}\int_{-1}^{1} Q_2(\tau, x)\,g_2(\tau)\,d\tau$$

The fact that the function $g_1(x)$ is odd, ensures the fulfilment of the condition of uniqueness of the displacements on traversing the circumference of the cut, given above in parentheses.

Substituting (12) into the second equation of (10), we obtain a singular, integro-differential equation

$$g_2(y) - \frac{\lambda\varkappa}{\pi}\int_{-1}^{1}\frac{g_2'(\tau)\,d\tau}{\tau - y} + \frac{1}{\pi}\int_{-1}^{1} G(\tau, y)\,g_2(\tau)\,d\tau = p_0(y) \tag{13}$$

$$G(\tau, y) = -\lambda Q_3(\tau, y) - \frac{e}{\pi^2}\int_{-1}^{1} R^+(\xi)\cos e\xi\, Q_2(\xi, \tau)\,d\xi \times \tag{14}$$

$$\int_{-1}^{1}\frac{Q_4(\eta, y)\,d\eta}{R^+(\eta)(\sin e\eta - \sin e\xi)}$$

$$p_0(y) = \frac{1}{\pi}\int_{-1}^{1}[R^+(\eta)]^{-1}Q_4(\eta, y)\sin e\eta\,d\eta - \frac{\lambda_0}{4\pi}$$

The stress intensity coefficient characterizing the singular character of the component of the stress field σ_y at the end $z = b + c$ of the cut l_0, is given by the expression

$$K = \lim_{z\to b+c} 2\sqrt{2\,|z-(b+c)|}\,\Phi(z)$$

which, by virtue of (8), (11) and (12), is reduced to the form

$$K = K_0\left(1 - e\pi^{-2}\sin^{-1} e\int_{-1}^{1} g_2(\tau)\,d\tau\int_{-1}^{1}\frac{R^+(\xi)\,Q_2(\xi,\tau)\cos e\xi}{\sin e\xi - \sin e}\,d\xi\right) \tag{15}$$

$$K_0 = p\,(^1/_2 ce^{-1}\operatorname{tg} e)^{1/2}$$

where K_0 denotes the value of this coefficient in the case when stringers are absent, and is obtained using the results of [5].

Let us also assess the effect of the cuts on the stress intensity at the ends of the stringers. We have

$$\tau_{xy} = \operatorname{Im}\,[\bar{z}\varphi''(z) + \psi'(z)] =$$
$$\operatorname{Im}\,[\varphi_{20}'(z) + \varphi_{40}'(z) + (z+\bar{z})\varphi_{20}''(z) + R_1(z)]$$

where $R_1(z)$ is a function continuous on L_0. According to (8) and the Sokhotskii formulas we have, on L_0,

$$\tau_{xy}^{+}(y) - \tau_{xy}^{-}(y) = -(1+\varkappa)\alpha_2'(y)$$

As we know [1], the tangential stresses $\tau_{xy}^{\pm}$ have a singularity of order $1/2$ at the end of the stringer lying at the finite part of the plane, and this implies that

$$\tau_{xy}^{\pm}(y) = \mp 1/2\,(1+\varkappa)(a^2-y^2)^{-1/2}h_1(y) + h_2(y)$$

where h_1 and h_2 are functions continuous on L_0.

We note that $\alpha_2(\pm a) = 0$ and this yields $g_2(1) = g_2(-1) = 0$. The latter relation is necessary for solving the equation (13). Indeed, according to (8) $\alpha_2(y)$ denotes the density of the potential functions $\varphi_{jk}(z)$ $(j = 2, 4)$ represented in the form of the Cauchy type integrals. The derivatives of these functions should have at the ends L_k singularities of the same order as τ_{xy}, i.e. of order $1/2$. But this implies that $\varphi_{jk}(z)$ are bounded at the points $z = \pm ia + 2kb$. In this case the assertions given in [8], Sect. 22 imply that $\alpha_2(y)$ vanishes at $y = \pm a$.

The stress intensity coefficient at the end of the stringer is

$$K_{st} = \lim_{y\to a}\sqrt{a-y}\,\tau_{xy}^{+}(y) = -\frac{1}{2}(\varkappa+1)\lim_{y\to a}\sqrt{a-y}\,\alpha_2'(y) = \frac{p\sqrt{a}}{2h}(\varkappa+1)\lim_{y\to 1}\frac{g_2(y)}{\sqrt{1-y^2}} \tag{16}$$

The integrals in (14) and (15) can be obtained in the form of convergent series in the powers of $r = \sin e$. This follows from the fact that the function $h(x, y)$ appearing in the kernels Q_2 and Q_4 can be expanded into a convergent series in powers of $\sin^2 ex / \mathrm{ch}^2 sy$, and from making the replacement $\sin ex = r\cos\theta$ which transforms the integrals into the standard table integrals. Following further the computing scheme given in [1], Sect. 11, we obtain a system of algebraic equations for (13).

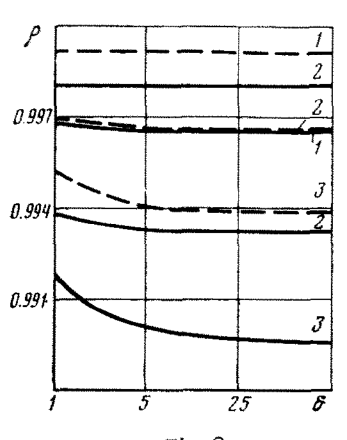

Fig. 2

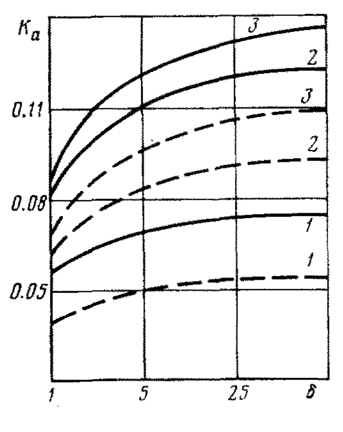

Fig. 3

Figure 2 depicts the coefficient $\rho = K / K_0$ versus the parameter $\delta = 2E_0S_0 / (Eah)$ at $\nu = 0.3$, for the values of c/b equal to 0.7 (dashed lines) and 0.9 (solid lines). The quantity a / b assumes the values of 0.4, 0.6 and 0.8 (corresponding to the lines 1—3). The effect of the stringer becomes appreciable when it is situated sufficiently near the cut end. When $c / b < 0.7$, the presence of stringers reduces the stress intensity by not more than 1%.

The same effect of removal of the stresses from the cut ends distributed along a straight line is observed when the stringers are replaced by weakenings in the form of transverse cuts oriented along the direction of the tensile force. We give, for comparison purposes, the values of the stress intensity coefficients for the cases of the system of cuts and stringers, and of a system of longitudinal-transverse cuts [4] (the corresponding quantities are denoted, respectively, by ρ_{st} and ρ_{sl} with $c / b = 0.7$ and the stringer stiffness parameter $\delta = 25$)

a/b	0.2	0.4	0.6	0.8
ρ_{st}	0.9996	0.9992	0.9967	0.9941
ρ_{sl}	0.9861	0.945	0.875	0.778

We must say however that, when a tensile force appears along the x-axis (Fig. 1) then the transverse cuts not only become the stress concentrators themselves, but they also lead to increase in stress along the longitudinal cuts.

The behavior of the stress intensity coefficient at the end of the stringer computed according to formula (16) is shown in Fig. 3 (with the parameters corresponding to the lines just as in Fig. 2) and here $K_a = K_{st}h\ [p\ (1 + \varkappa)]^{-1}$. The reciprocal effect of the crack on the stress intensity coefficient at the end of the stringer is also appreciable.

Increasing the rigidity of the stringers to the values exceeding $\delta = 25$, has no noticeable effect on the magnitude of the intensity coefficients neither at the cut ends, nor at the end of the stringer.

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